Elliptic Adventures of a Mathematician in Prison

Short Undergraduate Mathematics Seminar - 74

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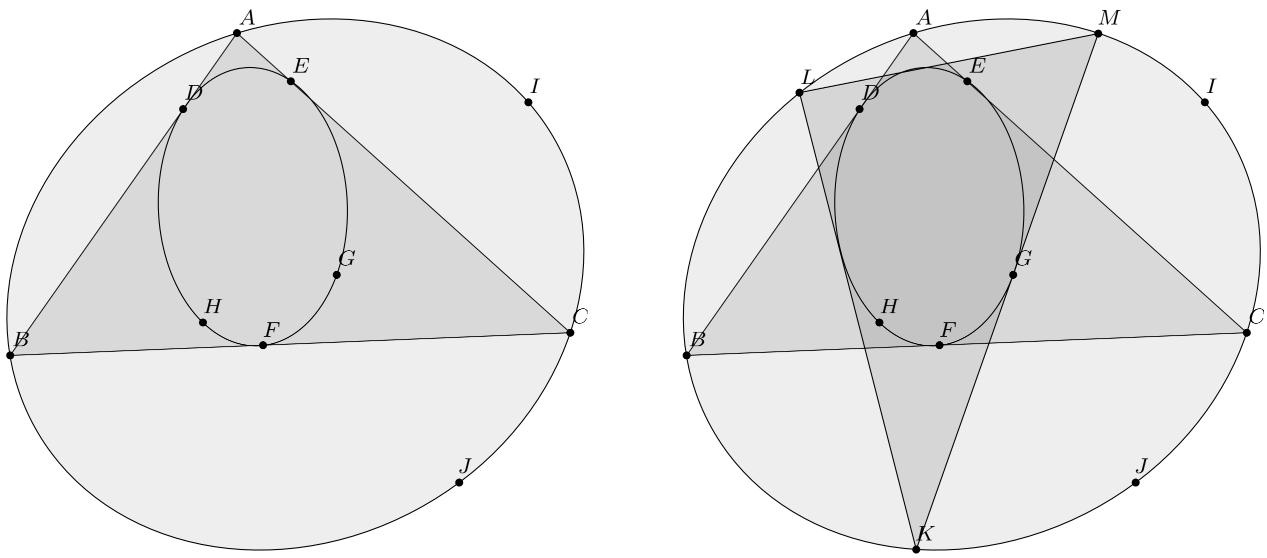
March 18, 2017

Abstract

This is seventh lecture in the series of ten lecture based on the lecture notes of Kyoto University (Japan) weekly mathematics seminars for high school students in December 1994 and January 1996[1]. The focus of these lectures is the interplay between topology, functions, geometry and algebra. This lecture is based on the lecture of Kenji Ueno[1], delivered in 1994, intended to be an introduction to *algebraic geometry*.

1 Motivation

You are given two ellipses[2], one lying inside the other. Suppose that you were able, through random trial and error, to draw a triangle inscribed in a given ellipse and circumscribed about the other given ellipse[3], as shown in Figure 1a.



(a) The two ellipses were found with respect to the given triangle by using appropriate 5 points to plot the conic.

(b) To draw this new triangle, we just need to make sure that its sides are tangent to the inner ellipse.

Figure 1: Triangles circumscribed and inscribed by ellipses. [Drawn using GeoGebra 4.0.34.0]

Now choose any point K on the outer ellipse and then draw a triangle which circumscribes the inner ellipse with K as one of its vertices, as shown in Figure 1b. Surprisingly, the ΔKLM that we have just drawn is completely inside the outer ellipse, and in fact it is inscribed in it. But there is nothing special about the point K, therefore if we are able to find one triangle circumscribing and inscribing the given pair of ellipses then there are infinitely many such triangles.

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[1]He is now the director of Yokkaichi University's Seki Kowa Institute of Mathematics.

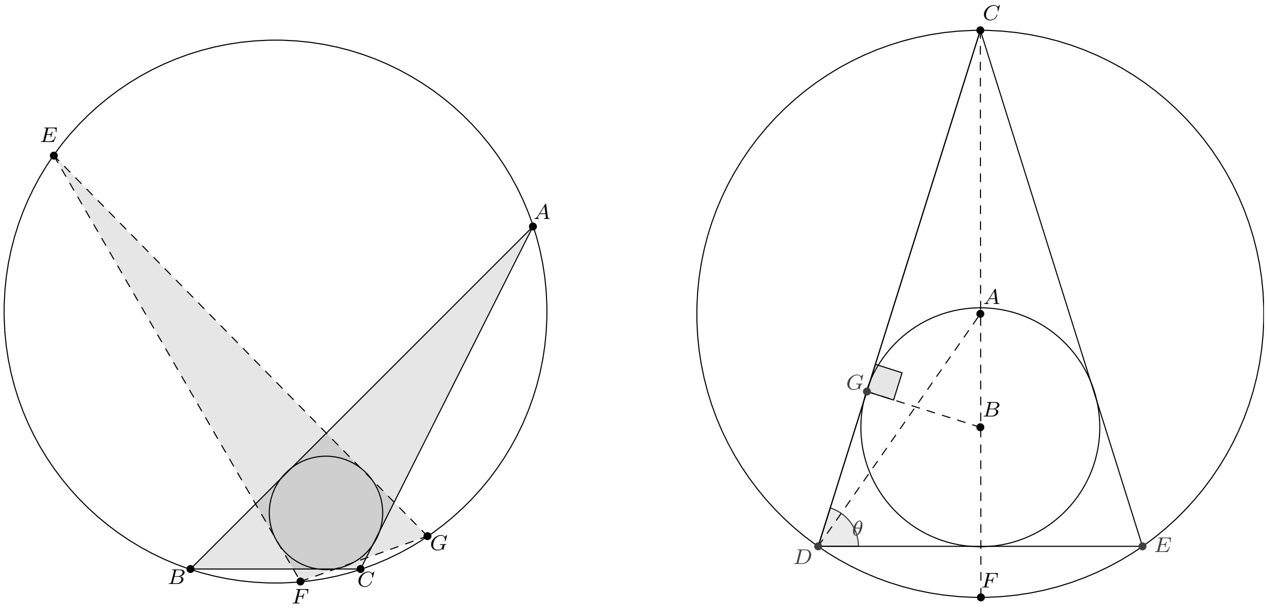
[2]We saw their formal definition in the third and sixth lecture of this series, SUMS-70 and SUMS-73.

[3]Unlike circles, there is no standard method to construct a triangle inscribed in an ellipse and circumscribed about another ellipse.

2 History

By substituting circles for the ellipses in the previous example, it becomes a simple matter of drawing a triangle and its circumscribed and inscribed circles. And this is how the idea of the theorem we wish to discuss in this lecture was born.

Theorem (Euler-Chapple Theorem)**.** *Let $\mathcal{C}$ be the circumscribed circle about a triangle ΔABC and $\mathcal{C}'$ the inscribed circle in the circle ΔABC. Let E be an arbitrary point on the outer circle $\mathcal{C}$. Then there exists a triangle with E as one of its vertices such that $\mathcal{C}$ is circumscribed about it and $\mathcal{C}'$ is inscribed in it.*



(a) First draw a triangle and then its incircle and circumcircle using standard straightedge-compass construction.

(b) Here $R = 5$ units, $d = 2$ units and $r = 2.1$ units, and the triangle is isosceles.

Figure 2: Triangle circumscribed and inscribed by circles. [Drawn using GeoGebra 4.0.34.0]

If R and r are the radii of $\mathcal{C}$ and $\mathcal{C}'$, respectively, and d is the distance between the centers of $\mathcal{C}$ and $\mathcal{C}'$, then $R^2 - d^2 = 2Rr$.

Remark. Since possibility of one triangle ensures the possibility of infinitely many triangles, it's sufficient to consider the simplest type of triangle to verify the formula once existence is established. The isosceles triangle as shown in Figure 2b (of which equilateral triangle is a special case) is perfect for this purpose since the trigonometric relations[4] can be used to conclude that

$$(R + d)\cos\theta = r; \quad \angle ADE = 2\theta - \pi/2; \quad R\sin(2\theta - \pi/2) = d + r = R(1 - 2\cos^2\theta)$$

Then eliminating the $\cos\theta$ from these set of equations we get the desired formula. ⋄

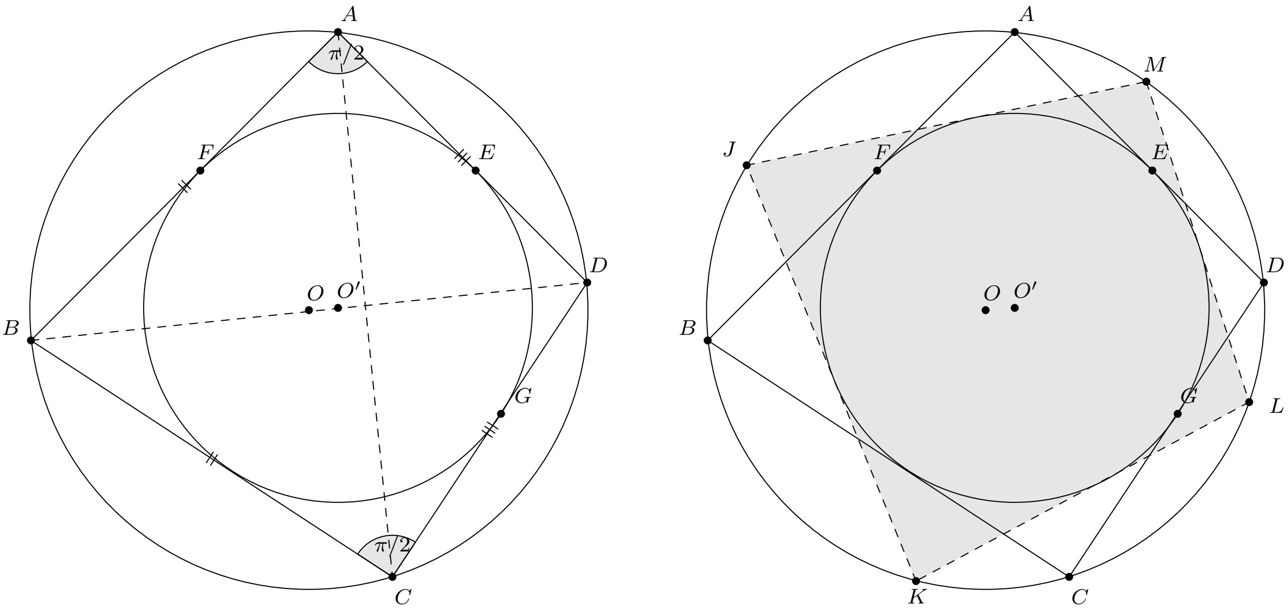
Euler used plane geometry to prove the theorem, using the concept of coaxial circles. This geometric proof is not simple, but relatively simpler proof by Ichiro Amemiya can be found in [1], pp. 117–126.

Euler-Chappel theorem is valid even if we change a triangle to a tetragon as indicated in the following theorem:

Theorem. *Given a tetragon/quadrilateral $ABCD$, suppose that there exist*[5] *the ciscumscribed circle $\mathcal{C}$ and the inscribed circle $\mathcal{C}'$. Then for any point J on $\mathcal{C}$, there exists a tetragon with J as one of the vertices such that $\mathcal{C}$ is the circumscribed circle about this tetragon and $\mathcal{C}'$ is the inscribed circle in this tetragon.*

[4] This is what we discussed in fifth lecture of this series, SUMS-72.

[5] In general, for a given tetragon, there is neither a circumscribed circle about it nor a circle inscribed in it. The quadrilaterals with circumcircle are called *cyclic quadrilaterals* and those with incircle are called *tangential quadrilateral*. Non-square rhombus is not cyclic and non-square rectangle is not tangential. If a quadrilateral is both cyclic and tangential then it is called *bicentric quadrilateral*.



(a) Start with a *right kite*, a kite with at least one right angle, since it's a well known bicentric quadrilateral. Then draw its incircle and circumcircle by using angle bisectors and perpendicular bisectors to locate their centers.

(b) To draw the new quadrilateral $JKLM$, we just need to make sure that its sides are tangent to the inner circle. By joining the point of intersection of these tangents with the outer circle, we get the desired quadrilateral.

Figure 3: Quadrilaterals circumscribed and inscribed by circles. [Drawn using GeoGebra 4.0.34.0]

If R and r the radii of the circumscribed circle $\mathcal{C}$ and inscribed circle $\mathcal{C}'$, respectively, and d is the distance between the centers of the circles $\mathcal{C}$ and $\mathcal{C}'$, that is the length of $\overline{OO'}$, then

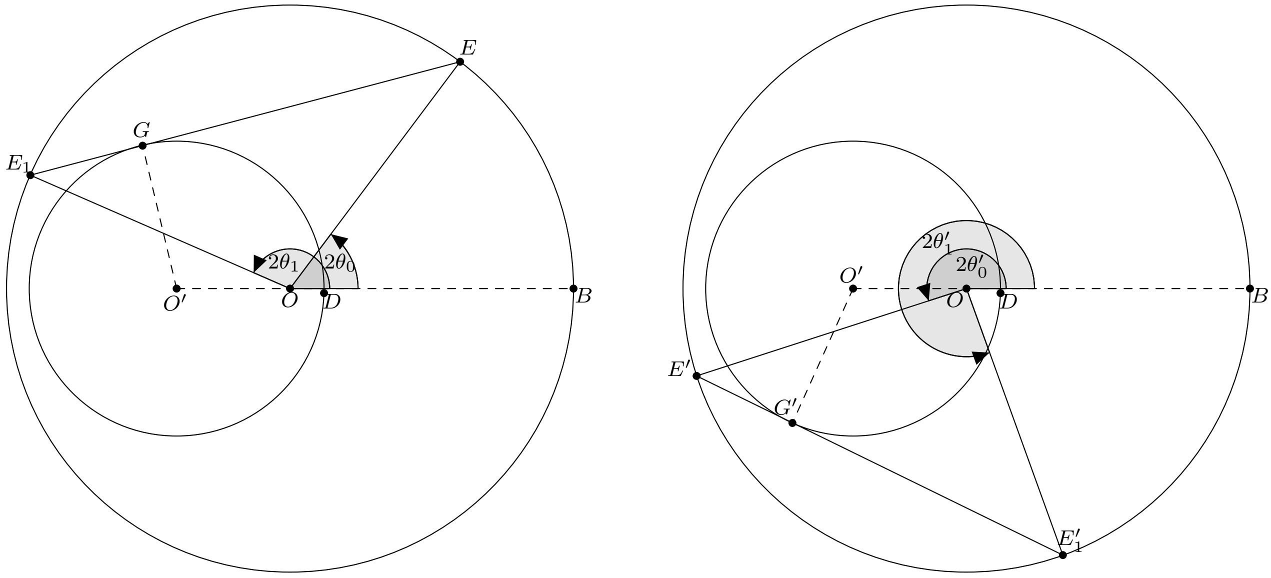
$$\frac{1}{(R+d)^2} + \frac{1}{(R-d)^2} = \frac{1}{r^2}$$

The equations in the above two theorems, describing the relation between R, r, and d (which indicates the relative locations of the circumscribed circle and the inscribed circle) is another example of the complexity and nonintuitive nature of these theorems.

Since in previous lecture of this series (SUMS-73) the concept of "Elliptic Functions" was introduced, let's have a look at how Carl Gustav Jacob Jacobi (1804-1851) proved this theorem by showing that the Poncelet theorem (stated later) for circles is somehow related to the elliptic integrals. Here is a simplified version of Jacobi's proof, given by Joseph Louis François Bertrand (1882-1900). In the case where circles are used in place of ellipses, we can prove the Poncelet theorem for any n-gon closure in circles, using the periodicity of elliptic integrals.

Theorem 1. *Given two circles $\mathcal{C}$ and $\mathcal{C}'$ where $\mathcal{C}'$ is located inside $\mathcal{C}$, choose a point E on $\mathcal{C}$. Starting at E, draw a line tangent to the inner circle $\mathcal{C}'$ and let E_1 be the point on $\mathcal{C}$ where this tangent line intersects the outer circle $\mathcal{C}$. From E_1 draw another line tangent to the inner circle $\mathcal{C}'$ and let E_2 be the point on $\mathcal{C}$ where the second tangent line intersects the outer circle $\mathcal{C}$. Repeat this process of drawing tangent lines to the inner circle $\mathcal{C}'$. Let $E_{n-1}E_n$ be the segment of the tangent line to the inner circle $\mathcal{C}'$ after n applications of this process. If $E_n = E$ for some $n \geq 3$, then for any point E on the circle $\mathcal{C}$, after n iterations of connecting points on $\mathcal{C}$ by lines tangent to $\mathcal{C}'$, the last intersection point will always be the same as the starting point.*

Proof. Consider two circles $\mathcal{C}$ and $\mathcal{C}'$ centered at O and O' with radii R and r, respectively (with $R > r$). Let d be the distance between the centers O and O' of these circles. Choose a point E on $\mathcal{C}$ and draw a tangent line to $\mathcal{C}'$ passing through E, with G being the point of tangency. This tangent line intersects $\mathcal{C}$ again at the point E_1 on $\mathcal{C}$. We also set $\angle BOE = 2\theta_0$ and $\angle BOE_1 = 2\theta_1$, as shown in Figure 4a. Similarly, we consider another tangent from point E' on $\mathcal{C}$ to the circle $\mathcal{C}'$. We mark angles as shown in Figure 4b.



(a) First example (b) Second example

Figure 4: Draw two circles, one contained inside other. Then draw a tangent to inner circle, and join the points of intersection of this tangent with the outer circle to its center. [Drawn using GeoGebra 4.0.34.0]

Jacobi discovered an astonishing fact about this figure[6] which enables us to make use of the periodicity of elliptic integral:

> If $k > 0$ is determined by the formula
>
> $$k^2 = \frac{4dR}{\sqrt{(R+d)^2 - r^2}}$$
>
> then the value of the elliptic integral
>
> $$J(\theta_0) = \int_{\theta_0}^{\theta_1} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}}$$
>
> is a real number which does not depend on the choice of θ_0, where θ_1 is determined by θ_0.

In other words, if we take another point E' on $\mathcal{C}$, and determine $2\theta'_0$ and $2\theta'_1$, then the value of the elliptic integral does not depend on the choice of E above, that is,

$$\int_{\theta_0}^{\theta_1} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} = \int_{\theta'_0}^{\theta'_1} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} = \omega$$

Suppose we draw tangent lines to the inner circle $\mathcal{C}'$ described in the theorem. If we repeat this process of drawing a tangent line ℓ times, we will obtain the sequence of points

$$E \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \ldots \longrightarrow E_\ell$$

Let $2\theta_0$ be the *parameter* corresponding to E, $2\theta_1$ to E_1, and so on until we have $2\theta_\ell$ corresponding to E_ℓ. Then

$$\begin{aligned} \int_{\theta_0}^{\theta_\ell} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} &= \int_{\theta_0}^{\theta_1} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} + \ldots + \int_{\theta_{\ell-1}}^{\theta_\ell} \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} \\ &= \underbrace{\omega + \ldots + \omega}_{\ell \textit{ times}} \\ &= \ell\omega \end{aligned}$$

[6] For proof see pp. 83–87 of [1].

If we repeat the process n times and the last point returns to the starting point so that $E_n = E$, we obtain

$$2\theta_n = 2\theta_0 + 2\pi$$

Hence, we will have the following relation:

$$\int_{\theta_0}^{\theta_0+\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_{\theta_0}^{\theta_n} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = n\omega \qquad (\star)$$

On the other hand, if the above relation holds, then we can say that $E = E_n$. That is, starting from E and drawing tangent lines to $\mathcal{C}$, n times, we go full circle and return to the starting point. Therefore, in order to prove the Poncelet theorem, we need to answer the following question.

> *If equation $(\star)$ is valid for some θ_0 is it also valid for any other θ ? That is, is the following statement true for any θ:*
>
> $$\int_{\theta}^{\theta+\pi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = n\omega \text{ ?}$$

The answer to this question is positive. This is because the integrand is a periodic function with period π. If a function is periodic, then the definite integral of that function over one period remains constant, no matter where the integration takes place. □

3 The Poncelet Closure Theorem

Generalization of the idea discussed in previous section was proved by Jean Victor Poncelet (1788-1867). He studied at the École Polytechnique in Paris. In November 1812, while serving as as a lieutenant of engineers in Napoleon's Russian campaign, Poncelet (pronounced poun-ce-le) was captured by the Russian army and became a prisoner of war. He was placed in a prison camp in Saratov, on the Volga River, where he spent the next two years. During these two years of imprisonment, he studied projective properties of geometric figures without the use of any books. During this time, he established the fundamental theory of projective geometry[7]. Poncelet's great work was published in Paris in 1822 as "Traité des propriétés projectives des figures". In sections 565-567 of this publication, you can find the theorem we will discuss in this section.

But, instead of original proof, we will discuss the proof given by Adolf Hurwitz (1859-1919) in 1878. The proof is based on the notion of algebraic correspondence.

Theorem 2 (The Poncelet Closure Theorem)**.** *Suppose that it is possible to construct an n-gon inscribed in an ellipse $\mathcal{C}$ and circumscribed about an ellipse $\mathcal{C}'$. Then, for any point P on $\mathcal{C}$, there exists an n-gon with P as one of the vertices which is inscribed in $\mathcal{C}$ and circumscribed about $\mathcal{C}'$.*

Proof. We can restate this theorem as Theorem 1, replacing "circle" by "ellipse":

> Given two ellipses $\mathcal{C}$ and $\mathcal{C}'$ where $\mathcal{C}'$ is located inside $\mathcal{C}$, choose a point P on $\mathcal{C}$. Starting at P, draw a line tangent to the inner ellipse $\mathcal{C}'$ and let P_1 be the point on $\mathcal{C}$ where this tangent line intersects the outer ellipse $\mathcal{C}$. From P_1 draw another line tangent to the inner ellipse $\mathcal{C}'$ and let P_2 be the point on $\mathcal{C}$ where the second tangent line intersects the outer ellipse $\mathcal{C}$. Repeat this process of drawing tangent lines to the inner ellipse $\mathcal{C}'$. Let $P_{n-1}P_n$ be the segment of the tangent line to the inner ellipse $\mathcal{C}'$ after n applications of this process. If $P_n = P$ for some $n \geq 3$, then for any point P on the ellipse $\mathcal{C}$, after n iterations of connecting points on $\mathcal{C}$ by lines tangent to $\mathcal{C}'$, the last intersection point will always be the same as the starting point.

Unlike the proof of Theorem 1, the proof of this theorem involves no calculus. In fact, the proof is completely algebraic. The foundation necessary to discuss the Poncelet theorem is that equations for

[7] I was introduced to projective geometry by Prof. S. A. Katre, on 15th May 2015. I was fascinated by the fact that there exists a plane where all lines intersect and for the first time questioned the idea of infinity in absolute sense. Then I spent a couple of weeks learning more about it from appendix of the book "Rational Points on Elliptic Curves" by Joseph H. Silverman.

ellipses are given by quadratic polynomials, and the knowledge of basic properties of quadratic equations. Note that ellipse is a curve satisfying the equations[8]

$$\Phi(x,y) = Ax^2 + By^2 + Cxy + Dx + Ey + F = 0; \quad \text{with } B^2 - 4AC < 0; \quad A, B, C, D, E, F \in \mathbb{R}$$

Since only the relative position of ellipses matter, we can move the whole system of two ellipses so as to establish the needed algebraic correspondence between the points P and P_1 on $\mathcal{C}$:

Step 1. Introduce the coordinate system by letting the origin be the center of the outer ellipse $\mathcal{C}$, the X-axis to be the line along the major axis, and the Y-axis to be the line along the minor axis. Then we derive following equation of the outer ellipse

$$\mathcal{C}: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; \quad \text{with} \quad 0 < b \leq a$$

Step 2. To prove the Poncelet theorem, we do not need to have an exact equation $\Phi(x,y)$ for inner ellipse $\mathcal{C}'$. The key point of this proof is the fact that *from each point on $\mathcal{C}$, we can draw two lines tangent to $\mathcal{C}'$*. So, if $\mathcal{C}'$ has its center at point (β, γ) and its major axis make an angle θ (anticlockwise) with x-axis, then we can use following transformation of coordinate system:

$$\text{Rotation} \begin{cases} \widetilde{x} = x\cos\theta + y\sin\theta \\ \widetilde{y} = -x\sin\theta + y\cos\theta \end{cases} \qquad \text{Translation} \begin{cases} \widetilde{x} = x - \beta \\ \widetilde{y} = y - \gamma \end{cases}$$

to get standardised equation of inner ellipse centered at origin given by

$$\widetilde{\mathcal{C}'}: \frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1; \quad \text{with} \quad 0 < b' \leq a'$$

Step 3. We can re-write the equations of the outer ellipse in parametric form[9] as:

$$\mathcal{C}: \begin{cases} x = f_1(s) = \frac{a(1-s^2)}{1+s^2} \\ y = f_2(s) = \frac{2bs}{1+s^2} \end{cases} ; \qquad -\infty < s < \infty$$

with point $(-a, 0)$ corresponding to the limit of the above parametric representation as s tends to $\pm\infty$.

My knowledge of *algebraic geometry* is limited to the first 10 chapters of the book "Algebraic Geometry for Scientists and Engineers" by Shreeram S. Abhyankar, which I read a couple of years ago. The book starts with following question:
"*Given an implicit equation, when can we obtain an explicit rational parametrization? Moreover, when can we even obtain a polynomial parametrization?*"
The answer to above question is stated in terms of following theorem:

Theorem. *A curve can be parametrized by polynomials if and only if it can be parametrized by rational functions and has only one point at infinity.*

From our knowledge of coordinate geometry, as stated above, we know that implicit equation of ellipse in standard form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has a rational parametrization.
Now for the conic section $Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$, the *points at infinity* are given by the terms of highest degree, that is, the two factors of the homogenous polynomial $Ax^2 + By^2 + Cxy$ of degree two will correspond to two points at infinity. In case of ellipse (and circle with $x^2 + y^2 = (x + iy)(x - iy)$) since $B^2 - 4AC < 0$, there are two points at infinity, both of which are complex. So, ellipse can't have a polynomial parametrization. In fact, parabola is the only conic section with polynomial parametrization.

[8] The equations used to plot outer and inner ellipse in Figure 1 were

$$\begin{cases} -1293.7035x^2 + 349.68656xy - 1528.85378y^2 + 23285.10518x - 6422.68799y = 73602.78283, \\ 209.70426x^2 + 8.86022xy + 97.84785y^2 - 3399.33534x - 126.36436y = -13198.74752 \end{cases}$$

[9] For proof using appropriate trigonometric substitutions see pp. 68–70 of [1].

Step 4. Now we will write parametric representation of inner ellipse $\mathcal{C}'$ by applying coordinate transformation to the parametric equation of $\widetilde{\mathcal{C}'}$ as

$$\widetilde{\mathcal{C}'} : \begin{cases} x = \frac{a'(1-t^2)}{1+t^2} \\ y = \frac{2b't}{1+t^2} \end{cases} \quad ; \qquad -\infty < t < \infty$$

with point $(-a', 0)$ corresponding to the limit of the above parametric representation as t tends to $\pm\infty$. Now by reversing the rotation and translation of coordinates applied in Step 2., we get

$$\mathcal{C}' : \begin{cases} x = g_1(t) = \frac{\widetilde{g_1}(t)}{1+t^2} \\ y = g_2(t) = \frac{\widetilde{g_2}(t)}{1+t^2} \end{cases} \quad ; \qquad -\infty < t < \infty$$

where both $\widetilde{g_1}(t)$ and $\widetilde{g_2}(t)$ are quadratic polynomials in t. Therefore, $\Phi(g_1(t), g_2(t)) = 0$ for all $t \in \mathbb{R}$.

Step 5. Now we wish to obtain the equation of tangent to $\mathcal{C}'$ drawn from some point P on $\mathcal{C}$. Let the point of tangency be $Q_1 \equiv (x_1, y_1)$ on $\mathcal{C}'$, then the equation of tangent is given by

$$\Phi_x(x_1, y_1)(x - x_1) + \Phi_y(x_1, y_1)(y - y_1) = 0$$

where $\Phi_x(x_1, y_1)$ and $\Phi_y(x_1, y_1)$ are the partial derivatives of Φ with respect to x and y, respectively, evaluated at the point (x_1, y_1).

This equation of tangent is generalization of the result we learned in the school about standard ellipse. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $a > b$ is the equation of ellipse, then the equation of tangent is given by

- $y = mx \pm \sqrt{a^2m^2 + b^2}$, if the slope m of tangent is given. (To derive this, assume that $y = mx + c$ is the equation of tangent and use it to eliminate y from the equation of ellipse; then you will get a quadratic equation in x whose discriminant should be zero for the given line to be tangent to given conic section. From there you get $c = \pm\sqrt{a^2m^2 + b^2}$.)

- $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$, if (x_1, y_1) is the point of tangency. (To derive this, note that the x-coordinate of the point of tangency is the solution of the quadratic equation obtained in deriving previous form of equation. Using this fact we find that $(\mp a^2m/c, \pm b^2/c)$ is the point of tangency; so just equate it to given point to get $m = -\frac{b^2x_1}{a^2y_1}$.)

From the second form of the equation of tangent, we observe that:
If $F(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$, then $F_x(x_1, y_1) = \frac{2x_1}{a^2}$ and $F_y(x_1, y_1) = \frac{2y_1}{b^2}$. Therefore, the equation of tangent at (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by

$$F_x(x_1, y_1)(x - x_1) + F_y(x_1, y_1)(y - y_1) = 0$$

Step 6. Let the points on the ellipses $\mathcal{C}$ and $\mathcal{C}'$ be expressed by parametric form stated in Step 3. and Step 4.. Then $P \equiv (f_1(s_0), f_2(s_0))$ and $Q_1 \equiv (g_1(t_1), g_2(t_1))$. Then from Step 5. we conclude that the equation of tangent from P to $\mathcal{C}'$ is given by

$$\Phi_x(g_1(t_1), g_2(t_1))(x - g_1(t_1)) + \Phi_y(g_1(t_1), g_2(t_1))(y - g_2(t_1)) = 0$$

Substituting the value of $\Phi_x(g_1(t_1), g_2(t_1))$ and $\Phi_y(g_1(t_1), g_2(t_1))$ we get

$$(2Ag_1(t_1) + Bg_2(t_1) + D)(x - g_1(t_1)) + (2Bg_1(t_1) + Cg_2(t_1) + E)(y - g_2(t_1)) = 0$$

Since, the tangent passes through P, we can put $(x, y) \equiv (f_1(s_0), f_2(s_0))$ to get

$$(2Ag_1(t_1) + Bg_2(t_1) + D)(f_1(s_0) - g_1(t_1)) + (2Bg_1(t_1) + Cg_2(t_1) + E)(f_2(s_0) - g_2(t_1)) = 0$$

Using Step 3. and Step 4., we can re-write this equation as

$$\frac{\delta(s_0, t_1)}{(1 + s_0^2)(1 + t_1^2)} = 0$$

where $\delta(s_0, t_1)$ is a polynomial of second degree in each of s_0 and t_1.

Step 7. Now our target is to get an equation involving the coordinates of $P \equiv (f_1(s_0), f_2(s_0)) \equiv P(s_0)$ and $P_1 \equiv (f_1(s_1), f_2(s_1)) \equiv P(s_1)$. Since the point $P \equiv (f_1(s_0), f_2(s_0))$ is an arbitrary point on $\mathcal{C}$, we can replace s_0 by s and t_1 by t in the equation obtained in Step 6.. Therefore, $\boxed{\delta(s, t) = 0}$, is an algebraic relation between s and t which expresses the existence of a tangent line to $\mathcal{C}'$ at $Q(t) \equiv (g_1(t), g_2(t))$ passing through $P(s) \equiv (f_1(s), f_2(s))$.

Therefore, if we view $\delta(s, t) = 0$ is a quadratic equation in s, then one of the solutions of this equations is s_0. The other solution s_1 corresponds to the point of intersection $P(s_1)$ of $\mathcal{C}$ with the line tangent to $\mathcal{C}'$ at $Q(t_1)$ passing through $P(s_0)$ on $\mathcal{C}$. Hence, we have an algebraic relation between $P(s_0)$ and $P(s_1)$ on $\mathcal{C}$ given by the *algebraic correspondence* via $Q(t_1)$ as:

$$\delta(s_0, t_1) = 0; \quad \delta(s_1, t_1) = 0$$

Eliminating t_1 from these two equations, we obtain the relation $\boxed{H(s_0, s_1) = 0}$ where $H(s_0, s_1)$ is an expression of second degree in each of s_0 and s_1. We know that $H(s_0, s_1)$ is a quadratic polynomial in s_1 because the equation $H(s_0, s) = 0$ in s always has two solutions s_1 and s_1'. In this example, s_1' is the value of the parameter for the intersection point on $\mathcal{C}$ with another tangent line to $\mathcal{C}'$ passing through $P(s_0)$. *We say that $H(s_0, s_1) = 0$ is the correspondence between $P(s_0)$ and $P(s_1)$ induced by the inner ellipse $\mathcal{C}'$.*

Note that we are working on *projective plane* and not *affine plane*. Projective plane is the union of affine plane and the set of directions in affine plane. Thus in projective plane there are no parallel lines. So, the case when $H(s_0, s_1) = 0$ has complex solutions is taken care of in this plane. For example, if the quadratic equation $H(s_0, s_1) = 0$ in s_1 had complex solutions (with s_0 fixed), this would mean that from $P(s_0)$ we can draw a tangent line to $\mathcal{C}'$ in the "complex direction" (in the sense of complex numbers). Actually, when we have concluded before that $H(s_0, s_1)$ is an expression of degree 2 in s_1, we implicitly assumed that we cannot draw a tangent line in the "imaginary direction".

Now, let us begin the actual proof! Take a point $P(s_0)$ of the ellipse $\mathcal{C}$ and draw the *counter-clockwise* tangent line to the ellipse $\mathcal{C}'$ passing through $P(s_0)$. This tangent line intersects $\mathcal{C}$ again at the point $P(s_1)$ on $\mathcal{C}$. Now, draw the tangent line to the ellipse $\mathcal{C}'$ passing through $P(s_1)$. This tangent line intersects $\mathcal{C}$ again at the point $P(s_2)$ on $\mathcal{C}$. With this process, we obtain algebraic relations given by polynomials of degree 2 in s_0 and s_1, and s_1 and s_2, $H(s_0, s_1) = 0$ and $H(s_1, s_2) = 0$.

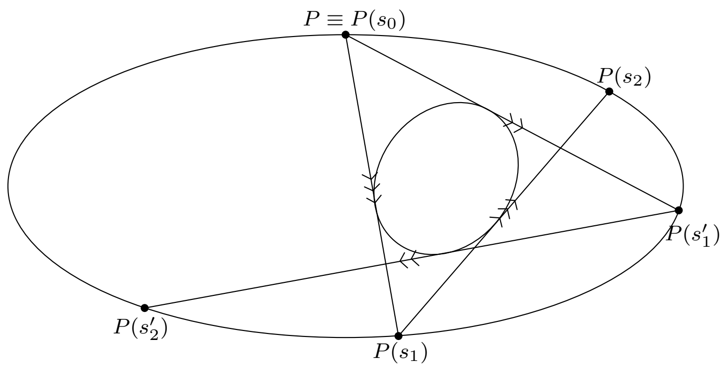


Figure 5: The two possible Poncelet sequence of correspondences. [Drawn using GeoGebra 4.0.34.0]

If we eliminate s_1 from the above two relations, we obtain an algebraic relation between s_0 and s_2

$$H_2(s_0, s_2) = 0$$

where $H_2(s_0, s_2)$ is a polynomial of degree 2 in each of s_0 and s_2 (Figure 5).

Considering the correspondence between two points on $\mathcal{C}$ in the manner described in the Poncelet theorem, we obtain a sequence of correspondences of points:

$$P(s_0) \longrightarrow P(s_1) \longrightarrow \ldots \longrightarrow P(s_k) \longrightarrow \ldots$$

where $\boxed{H(s_k, s_{k+1}) = 0}$ for all k. Hence we obtain the algebraic relation $\boxed{H_k(s_0, s_k) = 0}$ of degree 2 in each of s_0 and s_k by eliminating s_{k-1} from $H_{k-1}(s_0, s_{k-1}) = 0$ and $H(s_{k-1}, s_k) = 0$.

If we repeat the process n times ($n \geq 3$) and the last point returns to the starting point so that $P(s_n) = P(s_0) \equiv P$, then

$$P \equiv P(s_0) \longrightarrow P(s_1) \longrightarrow \ldots \longrightarrow P(s_{n-1}) \longrightarrow P(s_0) \equiv P$$

and the above algebraic correspondence is described by the relation

$$H_n(s_0, s_0) = 0$$

Note the cyclical nature of the sequence of correspondences. For example, if we start at the point $P(s_1)$, we have

$$P(s_1) \longrightarrow P(s_2) \longrightarrow \ldots \longrightarrow P(s_{n-1}) \longrightarrow P(s_0) \longrightarrow P(s_1)$$

where the last point corresponds to the starting point after n steps. Therefore, we have the relation $H_n(s_1, s_1) = 0$. So, in general we have

$$H_n(s_i, s_i) = 0 \qquad \text{for} \quad i = 0, 1, 2, \ldots, n-1$$

It we consider the above relations from an algebraic point of view, they show that the quadratic equation $\boxed{H_n(x, x) = 0}$ has n solutions $s_0, s_1, \ldots, s_{n-1}$. That is, the solution of $H_n(s_j, x) = 0$ are s_j obtained by the *counter-clockwise* construction of Poncelet sequence of correspondences

$$P(s_j) \longrightarrow P(s_{j+1}) \longrightarrow \ldots \longrightarrow P(s_{n-1}) \longrightarrow P(s_0) \longrightarrow P(s_1) \longrightarrow \ldots \longrightarrow P(s_j)$$

and the *clockwise* construction of Poncelet sequence of correspondences

$$P(s_j) \longrightarrow P(s_{j-1}) \longrightarrow \ldots \longrightarrow P(s_0) \longrightarrow P(s_{n-1}) \longrightarrow P(s_{n-2}) \longrightarrow \ldots \longrightarrow P(s_j)$$

Hence, the quadratic equation $\boxed{H_n(x, x) = 0}$ has $2n$ (≥ 6) solutions (counting the multiplicities). This can only happen when $H_n(x, x)$ is identically zero.

Therefore, for any point $\widetilde{P}(\widetilde{s})$ on $\mathcal{C}$ we have $\boxed{H_n(\widetilde{s}, \widetilde{s}) = 0}$. The geometric interpretation of the above relation is that if we start at an arbitrary point $\widetilde{P}$ and apply the operation of drawing a tangent line to $\mathcal{C}$, then we will return to the starting point $\widetilde{P}$.

We have completed the proof of the Poncelet closure theorem. □

Using the properties complex projective plane, one can generalize this theorem for any pair of conic sections.

Theorem. *Let $\mathcal{C}$ and $\mathcal{C}'$ are two plane conics. If it is possible to find, for a given $n \geq 3$, one n-sided polygon that is simultaneously inscribed in $\mathcal{C}$ and circumscribed around $\mathcal{C}'$, then it is possible to find infinitely many of them.*

For example, GeoGebra applet by Michael Borcherds, showing Poncelet's Closure Theorem for a general Ellipse and a Parabola: https://ggbm.at/c4gASMe9.

References

[1] Kenji Ueno, Koji Shiga and Shigeyuki Morita. *A Mathematical Gift, II: The interplay between topology, functions, geometry, and algebra*. The American Mathematical Society, 2004 (Translated from Japanese to English by Eiko Tyler)