Ulam spiral

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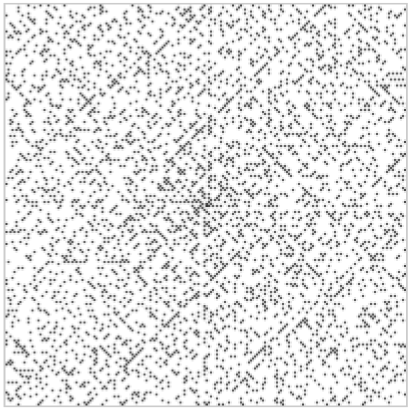
The **Ulam spiral** or **prime spiral** (in other languages also called the **Ulam cloth**) is a graphical depiction of the set of prime numbers, devised by mathematician Stanislaw Ulam in 1963 and popularized in Martin Gardner's *Mathematical Games* column in *Scientific American* a short time later.[1] It is constructed by writing the positive integers in a square spiral and specially marking the prime numbers.



Ulam spiral of size 200×200. Black dots represent prime numbers. Diagonal, vertical, and horizontal lines with a high density of prime numbers are clearly visible.

Ulam and Gardner emphasized the striking appearance in the spiral of prominent diagonal, horizontal, and vertical lines containing large numbers of primes. Both Ulam and Gardner noted that the existence of such prominent lines is not unexpected, as lines in the spiral correspond to quadratic polynomials, and certain such polynomials, such as Euler's prime-generating polynomial $x^2 - x + 41$, are believed to produce a high density of prime numbers.[2][3] Nevertheless, the Ulam spiral is connected with major unsolved problems in number theory such as Landau's problems. In particular, no quadratic polynomial has ever been proved to generate infinitely many primes, much less to have a high asymptotic density of them, although there is a well-supported conjecture as to what that asymptotic density should be.

In 1932, more than thirty years prior to Ulam's discovery, the herpetologist Laurence M. Klauber constructed a triangular, non-spiral array containing vertical and diagonal lines exhibiting a similar concentration of prime numbers. Like Ulam, Klauber noted the connection with prime-generating polynomials, such as Euler's.[4]

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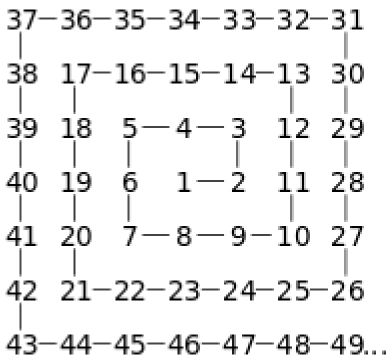
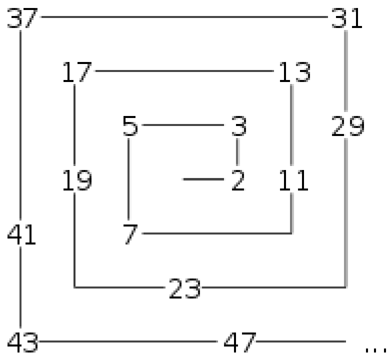
Construction

The **number spiral** is constructed by writing the positive integers in a spiral arrangement on a square lattice, as shown.



The Ulam spiral is produced by specially marking the prime numbers—for example by circling the primes or writing only the primes or by writing the prime numbers and non-prime numbers in different colors—to obtain a figure like the one below.



In the figure, primes appear to concentrate along certain diagonal lines. In the 200×200 Ulam spiral shown above, diagonal lines are clearly visible, confirming that the pattern continues. Horizontal and vertical lines with a high density of primes, while less prominent, are also evident. Most often, the number spiral is started with the number 1 at the center, but it is possible to start with any number, and the same concentration of primes along diagonal, horizontal, and vertical lines is observed. Starting with 41 at the center gives a particularly impressive example, with a diagonal containing an unbroken string of 40 primes, part of which is shown below.

296	295	294	293	292	291	290	289	288	287	286	285	284	283	282	281
237	236	235	234	233	232	231	230	229	228	227	226	225	224	223	280
238	185	184	183	182	181	180	179	178	177	176	175	174	173	222	279
239	186	141	140	139	138	137	136	135	134	133	132	131	172	221	278
240	187	142	105	104	103	102	101	100	99	98	97	130	171	220	277
241	188	143	106	77	76	75	74	73	72	71	96	129	170	219	276
242	189	144	107	78	57	56	55	54	53	70	95	128	169	218	275
243	190	145	108	79	58	45	44	43	52	69	94	127	168	217	274
244	191	146	109	80	59	46	41	42	51	68	93	126	167	216	273
245	192	147	110	81	60	47	48	49	50	67	92	125	166	215	272
246	193	148	111	82	61	62	63	64	65	66	91	124	165	214	271
247	194	149	112	83	84	85	86	87	88	89	90	123	164	213	270
248	195	150	113	114	115	116	117	118	119	120	121	122	163	212	269
249	196	151	152	153	154	155	156	157	158	159	160	161	162	211	268
250	197	198	199	200	201	202	203	204	205	206	207	208	209	210	267
251	252	253	254	255	256	257	258	259	260	261	262	263	264	265	266

History

According to Gardner, Ulam discovered the spiral in 1963 while doodling during the presentation of "a long and very boring paper" at a scientific meeting.[1] These hand calculations amounted to "a few hundred points". Shortly afterwards, Ulam, with collaborators Myron Stein and Mark Wells, used MANIAC II at Los Alamos Scientific Laboratory to extend the calculation to about 100,000 points. The group also computed the density of primes among numbers up to 10,000,000 along some of the prime-rich lines as well as along some of the prime-poor lines. Images of the spiral up to 65,000 points were displayed on "a scope attached to the machine" and then photographed.[5] The Ulam spiral was described in Martin Gardner's March 1964 *Mathematical Games* column in *Scientific American* and featured on the front cover of that issue. Some of the photographs of Stein, Ulam, and Wells were reproduced in the column.

In an addendum to the *Scientific American* column, Gardner mentioned the earlier paper of Klauber.[6][7] Klauber describes his construction as follows, "The integers are arranged in triangular order with 1 at the apex, the second line containing numbers 2 to 4, the third 5 to 9, and so forth. When the primes have been indicated, it is found that there are concentrations in certain vertical and diagonal lines, and amongst these the so-called Euler sequences with high concentrations of primes are discovered."[4]

Explanation

Diagonal, horizontal, and vertical lines in the number spiral correspond to polynomials of the form

$$f(n) = 4n^2 + bn + c$$

where b and c are integer constants. When b is even, the lines are diagonal, and either all numbers are odd, or all are even, depending on the value of c. It is therefore no surprise that all primes other than 2 line in alternate diagonals of the Ulam spiral. To understand why some odd diagonals have a higher concentration of primes than others, it is necessary to understand the behavior of the corresponding quadratic polynomials modulo odd primes.

Hardy and Littlewood's Conjecture F

In their 1923 paper on the Goldbach Conjecture, Hardy and Littlewood stated a series of conjectures, one of which, if true, would explain some of the striking features of the Ulam spiral. This conjecture, which Hardy and Littlewood called "Conjecture F", is a special case of the Bateman–Horn conjecture and asserts an asymptotic formula for the number of primes of the form $ax^2 + bx + c$. Rays emanating from the central region of the Ulam spiral making angles of 45° with the horizontal and vertical correspond to numbers of the form $4x^2 + bx + c$ with b even; horizontal and vertical rays correspond to numbers of the same form with b odd. Conjecture F provides a formula that can be used to estimate the density of primes along such rays. It implies that there will be considerable variability in the density along different rays. In particular, the density is highly sensitive to the discriminant of the polynomial, $b^2 - 16c$.

Conjecture F is concerned with polynomials of the form $ax^2 + bx + c$ where a, b, and c are integers and a is positive. If the coefficients contain a common factor greater than 1 or if the discriminant $\Delta = b^2 - 4ac$ is a perfect square, the polynomial factorizes and therefore produces composite numbers as x takes the values 0, 1, 2, ... (except possibly for one or two values of x where one of the factors equals 1). Moreover, if $a + b$ and c are both even, the polynomial produces only even values, and is therefore composite except possibly for the value 2. Hardy and Littlewood assert that, apart from these situations, $ax^2 + bx + c$ takes prime values infinitely often as x takes the values 0, 1, 2, ... This statement is a special case of an earlier conjecture of Bunyakovsky and remains open. Hardy and Littlewood further assert that, asymptotically, the number $P(n)$ of primes of the form $ax^2 + bx + c$ and less than n is given by

$$P(n) \sim A \frac{1}{\sqrt{a}} \frac{\sqrt{n}}{\log n}$$

where A depends on a, b, and c but not on n. By the prime number theorem, this formula with A set equal to one is the asymptotic number of primes less than n expected in a random set of numbers having the same density as the set of numbers of the form $ax^2 + bx + c$. But since A can take values bigger or smaller than 1, some polynomials, according to the conjecture, will be especially rich in primes, and others especially poor. An unusually rich polynomial is $4x^2 - 2x + 41$ which forms a visible line in the Ulam spiral. The constant A for this polynomial is approximately 6.6, meaning that the numbers it generates are almost seven times as likely to be prime as random numbers of comparable size, according to the conjecture. This particular polynomial is related to Euler's prime-generating polynomial $x^2 - x + 41$ by replacing x with $2x$, or equivalently, by restricting x to the even numbers. Hardy and Littlewood's formula for the constant A is

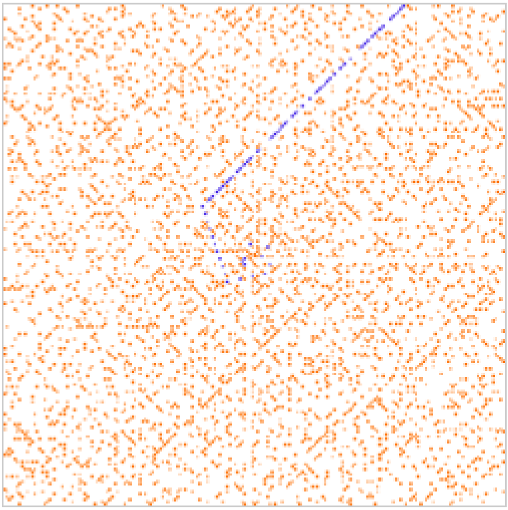
$$A = \varepsilon \prod_p \left(\frac{p}{p-1}\right) \prod_\varpi \left(1 - \frac{1}{\varpi - 1}\left(\frac{\Delta}{\varpi}\right)\right).$$

A simpler, but obviously equivalent formula is given by:

$$A = \prod_p \frac{1 - \frac{\omega(p)}{p}}{1 - \frac{1}{p}}$$

, where p runs over all primes, and $\omega(p)$ — is number of zeros of the quadratic polynomials modulus 'p'.

It further simplifies to $\prod_p \frac{p - \omega(p)}{p-1}$.



The primes of the form $4x^2 - 2x + 41$ with $x = 0, 1, 2, ...$ have been highlighted in purple. The prominent parallel line in the lower half of the figure corresponds to $4x^2 + 2x + 41$ or, equivalently, to negative values of x.

In the first formula, explanation is a little bit more complex. There, in the first product, p is an odd prime dividing both a and b; in the second product, ϖ is an odd prime not dividing a. The quantity ε is defined to be 1 if $a + b$ is odd and 2 if $a + b$ is even. The symbol $\left(\frac{\Delta}{\varpi}\right)$ is the Legendre symbol. A quadratic polynomial with $A \approx 11.3$, currently the highest known value, has been discovered by Jacobson and Williams.[8][9]

Variants

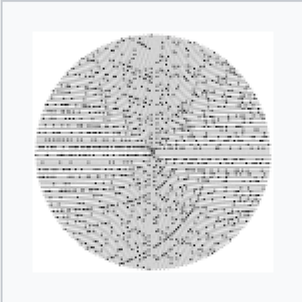
Klauber's 1932 paper describes a triangle in which row n contains the numbers $(n - 1)^2 + 1$ through n^2. As in the Ulam spiral, quadratic polynomials generate numbers that lie in straight lines. Vertical lines correspond to numbers of the form $k^2 - k + M$. Vertical and diagonal lines with a high density of prime numbers are evident in the figure.

Robert Sacks devised a variant of the Ulam spiral in 1994. In the Sacks spiral, the non-negative integers are plotted on an Archimedean spiral rather than the square spiral used by Ulam, and are spaced so that one perfect square occurs in each full rotation. (In the Ulam spiral, two squares occur in each rotation.) Euler's prime-generating polynomial, $x^2 - x + 41$, now appears as a single curve as x takes the values 0, 1, 2, ... This curve asymptotically approaches a horizontal line in the left half of the figure. (In the Ulam spiral, Euler's polynomial forms two diagonal lines, one in the top half of the figure, corresponding to even values of x in the sequence, the other in the bottom half of the figure corresponding to odd values of x in the sequence.)

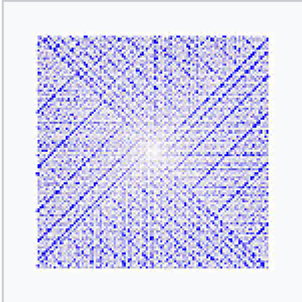
Additional structure may be seen when composite numbers are also included in the Ulam spiral. The number 1 has only a single factor, itself; each prime number has two factors, itself and 1; composite numbers are divisible by at least three different factors. Using the size of the dot representing an integer to indicate the number of factors and coloring prime numbers red and composite numbers blue produces the figure shown.

Spirals following other tilings of the plane also generate lines rich in prime numbers, for example hexagonal spirals.

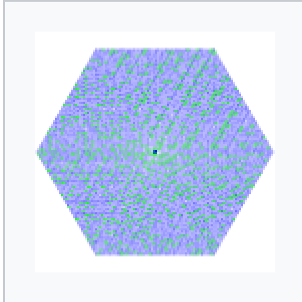
Klauber triangle with prime numbers generated by Euler's polynomial $x^2 - x + 41$ highlighted.



Sacks spiral.



Ulam spiral of size 150×150 showing both prime and composite numbers.



Hexagonal number spiral with prime numbers in green and more highly composite numbers in darker shades of blue.

Number spiral with 7503 primes visible on regular triangle.

See also

- Pattern recognition
- Prime k-tuple

References

1. Gardner 1964, p. 122.
2. Stein, Ulam & Wells 1964, p. 517.
3. Gardner 1964, p. 124.
4. Daus 1932, p. 373.
5. Stein, Ulam & Wells 1964, p. 520.
6. Gardner 1971, p. 88.
7. Hartwig, Daniel (2013), *Guide to the Martin Gardner papers* (http://www.oac.cdlib.org/findaid/ark:/13030/kt6s20356s/), The Online Archive of California, p. 17.
8. Jacobson Jr., M. J.; Williams, H. C (2003), "New quadratic polynomials with high densities of prime values", *Mathematics of Computation*, **72** (241): 499–519, doi:10.1090/S0025-5718-02-01418-7 (https://doi.org/10.1090%2FS0025-5718-02-01418-7)
9. Guy, Richard K. (2004), *Unsolved problems in number theory* (https://books.google.com/?id=1AP2CEGxTkgC) (3rd ed.), Springer, p. 8, ISBN 978-0-387-20860-2

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External links

- Prime Spirals - Numberphile, YouTube video with Dr. James Grime and the University of Nottingham
- 41 and more Ulam's Spiral - Numberphile, YouTube video with Dr. James Clewett and the University of Nottingham
- Ulam spiral, Interactive Javascript application that displays the prime numbers up to 10^{18}.

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